A Hybrid Approach to Nonlinear System Stability and Performance

The problems of stability and performance for nonlinear feedback systems are examined. A hybrid approach that combines input-output and state-space theories is proposed. Within this framework the notions of dynamic gain and dynamic incremental gain of a nonlinear operator are reexamined and their inability to accurately reflect the dynamic behavior of a physical system is demonstrated. The concepts of dynamic gain and dynamic incremental gain *over sets* are introduced and their connection to a physical system's dynamic behavior is established. Theorems are developed that allow the systematic computation of these gains through nonsmooth optimal control theory. Illustrative examples are provided throughout the paper.

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Introduction

Control problems pose two challenges for the designer: a *stability problem* to guarantee that the closed-loop system will not let the system variables drift away from their desired values, and a *performance problem* to identify and implement a control strategy that maintains the system variables as close to their desired values as possible.

Through the years, system theory has been approached via two distinct pathways: the state-space and input-output theories.

State-space theory, pioneered by Liapunov and Poincare, is based on the realization of a system by a set of first-order differential equations and their corresponding initial conditions. The notion of stability is quantified by examining the time evolution of the system states when their initial values are perturbed. The notion of performance is quantified by a cost functional defined on the state and input spaces. A recent success of that approach is the exact linearization of nonlinear feedback systems in various versions (Hunt et al., 1983, 1986; Isidori and Ruberti, 1984; Kantor, 1986; Kravaris and Chung, 1987).

Input-output theory, first applied by Sandberg (1964) and Zames (1966a, b) does not examine the intrinsic organization of the system but rather focuses on its relations with its surroundings. Inputs and outputs are the fundamental variables in this formulation, whereas the states, if used, are merely auxiliary variables. The system is then regarded as a mapping between two normed spaces consisting of the input and output signals respectively. Input-output stability is identified with the boundedness of the input-output map, whereas performance is quantified by the norm of an appropriately chosen input-output map.

Although both approaches can be used for control purposes, input-output theory provides several advantages. In recent years many significant developments have taken place in linear input-output theory [characterization of all stabilizing controllers for a given plant (Youla et al., 1974), H_∞ optimization (Francis et al., 1984), μ analysis and synthesis (Doyle, 1984)]. However, the theory's applicability to nonlinear systems has been restricted. A frequent obstacle is that there is no systematic way to compute the norm of a stable nonlinear operator, despite the fact that this is necessary in tasks such as:

a. Quantification of open and closed loop stability for nonlinear systems. Stability theorems (e.g., small gain theorem) would then be possible to verify and use in practice.

b. L_p optimization for nonlinear systems.

c. Use of the Q parametrization method for nonlinear systems (Desoer and Liu, 1982) which hinges on the incremental stability of the plant to be controlled.

d. Assessment of the closeness of two nonlinear operators. For example, the approximation of a nonlinear operator with fading memory by one linear in dynamics and nonlinear in the outputs (Boyd and Chua, 1985) could then be investigated.

In this paper we propose a *hybrid* approach that combines both input-output and state-space theories. Within this framework, we examine the standard notions of gain and incremental gain of an operator and expose their shortcomings in assessing a system's stability and performance characteristics. Indeed, we establish that the notion of *nonlinear operator norm* is too

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restrictive and sometimes misleading, since it refers to signals taking values in the entire interval $(-\infty, \infty)$, while physical magnitudes are usually constrained in subsets of $(-\infty, \infty)$ (e.g., a flow rate can take values from zero to a certain maximum, as the corresponding valve changes from completely closed to completely open, respectively). To remedy these shortcomings we introduce the novel concepts of *gains* and *incremental gains over sets*, which realistically reflect a system's stability and performance characteristics. Within the hybrid framework direct computations of *gains* and *incremental gains over sets* become feasible, thus providing invaluable insight into a system's behavior and guidelines for the design of an effective control system.

The remaining paper is structured as follows: In the following section we provide mathematical background on input-output theory and nonsmooth optimal control. Next, we establish that the existing input-output stability notions may be misleading, we propose a new formulation of input-output stability for dynamical systems and introduce the notions of gains and incremental gains over sets. Finally, we examine the performance problem.

Mathematical Background

Input-output theory

The input-output system formulation (familiar to process control engineers as the transfer function approach) has been extended to nonlinear systems in Zames (1966a, b), Willems (1971), and Desoer and Vidyasagar (1975).

Within this framework the magnitude of a signal x: $[0, \infty) \rightarrow R^n$ is quantified through its *p norm*, defined as

$$\|x\|_p \triangleq \begin{cases} \left[\int_{S\subset R} \|x(t)\|^p \, dt\right]^{1/p} & \text{if } p \in [1, \infty) \\ \sup_{t\in[0,\infty)} \|x(t)\| & \text{if } p = \infty \end{cases} \tag{1}$$

where $\|x(t)\|$ denotes any norm of the vector $x(t)$.

Signals with finite p norms form a (Banach) space L_p^n, defined as

$$L_p^n \triangleq \{x\colon [0, \infty) \rightarrow R^n\colon \|x\|_p < \infty\} \tag{2}$$

Thus a signal belongs to L_∞^n if its value is bounded for all $t \geq 0$.

A physical signal always has a finite value. Thus if one assigns to the physical signal x a *truncated* signal

$$x_T\colon t \rightarrow x_T(t) = \begin{cases} x(t) & \text{if } t \leq T \\ 0 & \text{if } t > T \end{cases} \tag{3}$$

then $\|x_T\|_p$ will be finite. This naturally leads to the definition of an *extended* (Banach) space

$$L_{pe}^n \triangleq \{x\colon [0, \infty) \rightarrow R^n\colon x_T \in L_p^n \quad \text{for all} \quad T \geq 0\} \tag{4}$$

The *dynamic* behavior of a nonlinear system is described by an unbiased nonlinear operator

$$N\colon L_{pe}^m \rightarrow L_{pe}^n\colon u \rightarrow y\colon 0 \rightarrow N(0) = 0 \tag{5}$$

which maps input signals u to output signals y. N is commonly realized through a set of differential (state) and algebraic (output) equations.

The (*dynamic*) *p gain* and the (*dynamic*) *incremental p gain* of N are defined, respectively, as

$$\|N\|_p \triangleq \sup_{u,T} \left\{\frac{\|Nu_T\|_p}{\|u_T\|_p} : u \in L_{pe}^m - \{0\}, T \geq 0\right\}$$

$$\|N\|_{\Delta p} \triangleq \sup_{u_1,u_2,T} \left\{\frac{\|Nu_{1T} - Nu_{2T}\|_p}{\|u_{1T} - u_{2T}\|_p} : u_1, u_2 \in L_{pe}^m, u_1 \neq u_2, T \geq 0\right\} \tag{6}$$

It is clear that $T_1 > T_2$ implies

$$\sup_u \frac{\|Nu_{T_1}\|_p}{\|u_{T_1}\|_p} \geq \sup_u \frac{\|Nu_{T_2}\|_p}{\|u_{T_2}\|_p},$$

since the set of input signals that vanish after $t = T_1$ strictly contains all input signals that vanish after $t = T_2 < T_1$. Therefore if one restricts the domain of N to signals in L_p^n, it holds that

$$\|N\|_p = \sup_u \left\{\frac{\|Nu\|_p}{\|u\|_p} : u \in L_p^m - \{0\}\right\} = \lim_{T\rightarrow\infty} \sup_u \left\{\frac{\|Nu_T\|_p}{\|u_T\|_p} : u \in L_{pe}^m - \{0\}\right\} \tag{7}$$

and

$$\|N\|_{\Delta p} = \sup_{u_1,u_2} \left\{\frac{\|Nu_1 - Nu_2\|_p}{\|u_1 - u_2\|_p} : u_1, u_2 \in L_p^m, u_1 \neq u_2\right\} = \lim_{T\rightarrow\infty} \sup_{u_1,u_2} \left\{\frac{\|Nu_{1T} - Nu_{2T}\|_p}{\|u_{1T} - u_{2T}\|_p} : u_1, u_2 \in L_{pe}^m, u_1 \neq u_2\right\}$$

Finally, the *linearization* of an operator $N\colon L_{pe}^m \rightarrow L_{pe}^n$ around the trajectory $u_o \in L_{pe}^m$ is a linear operator $L_{u_o}\colon L_{pe}^m \rightarrow L_{pe}^n$ such that

$$\lim_{\|u\|_p\rightarrow 0} \frac{\|N(u_o + u) - Nu_o - L_{u_o}u\|_p}{\|u\|_p} = 0 \tag{8}$$

Nonsmooth calculus and nonsmooth optimal control

In the classical optimal control problem (Athans and Falb, 1966) solved by Pontryagin (1962), certain differentiability assumptions are made with regard to the functions involved. Later in this article we demonstrate the interrelation between the input-output properties of a system and optimal control problems. Since these problems involve nondifferentiable functions, we present elements of nonsmooth calculus in Appendix A and formulate next the nonsmooth optimal control problem. Loosely speaking, the solution to this problem follows the "maximum principle" pattern, the difference being that for nondifferentiable functions *generalized gradients* take the role of *derivatives*. For a very detailed formulation the reader is referred to Athans and Falb (1966) for the smooth and Clarke (1983) for the nonsmooth case.

The nonsmooth optimal control problem is as follows:

Let $\quad L\colon R^n \times R^m \rightarrow R\colon (x(t), u(t)) \rightarrow L(x(t), u(t))$

and

$$f\colon \boldsymbol{R}^n \rightarrow \boldsymbol{R}\colon x(t_1) \rightarrow f(x(t_1))$$

$$\phi\colon \boldsymbol{R}^n \times \boldsymbol{R}^m \rightarrow \boldsymbol{R}^n\colon (x(t), u(t)) \rightarrow \phi(x(t), u(t))$$

where $L(\cdot, u(t)), f(\cdot)$, and $\phi(\cdot, u(t))$ are locally Lipschitz of rank $\kappa_L(u(t))$, κ_f, and $\kappa_\phi(u(t))$, respectively. Also consider

i) $\dot{x}(t) = \phi(x(t), u(t))$ on $(0, t_1]$
ii) $x(0) \in C_o \subset \boldsymbol{R}^n, x(t_1) \in C_1 \subset \boldsymbol{R}^n, C_o, C_1$ closed (but not necessarily bounded)
iii) $u \in \Omega = \{u\colon \boldsymbol{R} \rightarrow \boldsymbol{R}^m\colon u(t) \in U(t) \subset \boldsymbol{R}^m\}$
Under these assumptions find:

$$\min_{u \in \Omega} \left[f(x(t_1)) + \int_0^{t_1} L(x(t), u(t))\, dt \right] \qquad (9)$$

Remark. No differentiability assumptions are made with respect to L, f, and ϕ. If, however, we ask that $\partial L/\partial x$, $\partial f/\partial x$, and $\partial \phi/\partial x$ exist and be continuous then the above problem reduces to the classical optimal control problem.

Fact (Clarke, 1983). Let u solve the nonsmooth optimal control problem. Then there exists a scalar λ equal to 0 or 1, an arc p: $(0, t_1] \rightarrow \boldsymbol{R}^n$ and a (***Hamiltonian***) function

$$H(x(t), p(t), u(t)) \triangleq \lambda L(x(t), u(t)) + \langle p(t), \phi(x(t), u(t)) \rangle \qquad (10)$$

such that:

i) Maximum (minimum) principle:

$$H(x(t), p(t), u(t)) = \min_{w \in U(t)} H(x(t), p(t), w) \quad \forall t \in (0, t_1] \qquad (11)$$

ii) Adjoint "equation":

$$-\dot{p}(t) \in \partial_x H(x(t), p(t), u(t)) \quad \forall t \in (0, t_1] \qquad (12)$$

iii) Transversality conditions: for some $\xi \in \partial f(x(t_1))$ it holds

$$\lambda \xi - p(t_1) \in N_{C_1} x(t_1), p(0) \in N_{C_o} x(0) \qquad (13)$$

iv) Normality condition

$$\|p\|_p + \lambda > 0 \qquad (14)$$

Remarks.

- If $C_1 = R^n$ (free end-point problem) and $f = 0$, then Eq. 13 yields $p(t_1) = 0$
- If H is differentiable with respect to $x(t)$, then Eq. 12 becomes

$$\dot{p}(t) = -\frac{\partial H}{\partial x}(x(t), p(t), u(t)) = -\lambda \left(\frac{\partial L}{\partial x}\right)^T - \left(\frac{\partial \phi}{\partial x}\right)^T p(t) \qquad (15)$$

- If the problem is not singular, i.e., no nonzero $p(t)$ exists satisfying $\dot{p}(t) = -[\partial \phi/\partial x]^T p(t)$, then $\lambda = 1$ is sufficient to find a solution.

Input-Output Stability

Input-output stability and its limitations

Stability is a focal issue in the study of dynamical systems. To quantify that notion the following definition of input-output stability is often given:

Definition. A system represented by the operator N (Eq. 5) is said to be L_p-stable if

$$u \in L_p^m \Longrightarrow Nu \in L_p^n \;\forall\, u \in L_p^m \qquad (16)$$

Qualitatively, this definition states that a system is stable if p-integrable inputs yield p-integrable outputs (for $1 \leq p < \infty$) or if bounded inputs yield bounded outputs (for $p = \infty$).

A more restrictive definition of L_p stability for the system represented by Eq. 5 requires that

$$\|N\|_p < \infty \qquad (17)$$

with $\|N\|_p$ defined as in Eq. 6. This requirement guarantees that every input signal $u \in L_p^m$ is mapped to an output signal $y \triangleq Nu \in L_p^n$ since, by Eq. 7, $\|Nu\|_p \leq \|N\|_p \cdot \|u\|_p$. It also implies that the amplification of any input signal is finite.

Remark. For linear systems defined through a superposition integral of the form

$$y(t) = \int_0^t W(t, \tau) u(\tau)\, d\tau$$

Eqs. 16 and 17 coincide. In fact, explicit formulae exist that provide $\|N\|_p$ for linear time-varying systems ($p = 1, p = \infty$) and for linear time-invariant systems ($p = 1, p = 2, p = \infty$) (Desoer and Vidyasagar, 1973).

Although the above definitions are mathematically sound, the following examples provide motivation for reexamining the scope of the above notions and redefining their range of applicability.

Example 1. Consider a continuous stirred tank heater with a heating coil and constant volume V. The heater is modeled by the dynamic and output equations

$$\dot{T}(t) = \frac{1}{V}\left[(F_s + \Delta F(t))(T_i - T(t)) + \frac{UA(T_c - T(t))}{\rho c_p}\right],$$

$$T(0) = T_s = T_i + \frac{UA/\rho c_p}{F_s + UA/\rho c_p}(T_c - T_i)$$

$$\Delta T(t) = T(t) - T_s \qquad (18)$$

where $T = T_s + \Delta T$ and $F = F_s + \Delta F$ are the controlled and manipulated variables, respectively. Then Eqs. 18 define an unbiased operator

$$N\colon L_{\infty e} \rightarrow L_{\infty e}\colon \Delta F \rightarrow \Delta T$$

Using the theorem of Appendix B and the minimum principle of optimal control we show in Appendix C that

$$\|N\|_\infty = \sup_{\Delta F \in L_\infty} \frac{\sup_{t\geq 0} |\Delta T(t)|}{\sup_{t \geq 0} |\Delta F(t)|} = \infty$$

Hence, according to the definition in Eq. 17 the system described by Eq. 18 is *unstable*.

This conclusion is of little value, since it refers to a situation which cannot physically occur. As Eq. C17 indicates, $\|N\|_\infty$ is infinity if

$$\Delta F(t) = -F_s - \frac{UA}{\rho c_p} \Longrightarrow F(t) = -\frac{UA}{\rho c_p} \quad \text{for all} \quad t \geq 0.$$

This suggests that a *negative* flow rate would result in instability. Since the flow rate is always *positive* this stability warning is of little value. From the previous discussion it becomes apparent that $\|N\|_p$, as defined in Eq. 7, is not an adequate stability indicator. Physical signals are constrained in *subsets* of L_∞^n and should be considered as such.

Example 2. Consider a nonisothermal CSTR modeled by the dynamic and output equations (Stephanopoulos, 1984)

$$\frac{dC_A(t)}{dt} = \frac{F}{V}(C_{Ai} - C_A(t)) - C_A(t)\alpha e^{-E/RT(t)}$$

$$\frac{dT(t)}{dt} = \frac{F}{V}(T_i - T(t)) - \left(\frac{\Delta H_R}{c_p \rho}\right) C_A(t)\alpha e^{-E/RT(t)} - \frac{Q_s + \Delta Q(t)}{c_p V \rho}$$

$$\Delta T(t) = T(t) - T_s, \quad \Delta C_A(t) = C_A(t) - C_{As} \tag{19}$$

and define the operator

$$N: \Delta Q \longrightarrow (\Delta T, \Delta C_A)$$

Also consider that the system operates at the upper of three steady states (Figure 1). Then, according to definition in Eq. 16 the system is "L_∞-stable," since all bounded inputs result in bounded outputs, while it is "L_p-unstable," for all $p \in [1, \infty)$, since there exist eventually decaying input signals that can perturb the system so that it reaches the lower steady state. This broad stability characterization, although not captured by analysis of locally linearized model dynamics, cannot quantify the stability domain associated with the steady state A.

Input-output stability over sets

The preceding examples have demonstrated that the notions of stability, as defined in the literature, can be misleading and may not accurately reflect the dynamic properties of a system. Next, we define dynamic stability within its natural setting, namely over sets of physically meaningful input signals.

Definition. A system $N: L_{pe}^m \longrightarrow L_{pe}^n$ is *stable* over the set $W \subseteq L_p^m$ if for every piecewise continuous $u \in W$ we have that $\|Nu\|_p < \infty, p \in [1, \infty)$.

In the above definition u is not simply required to belong to L_p^m but rather to be contained in a set W which appears explicitly in

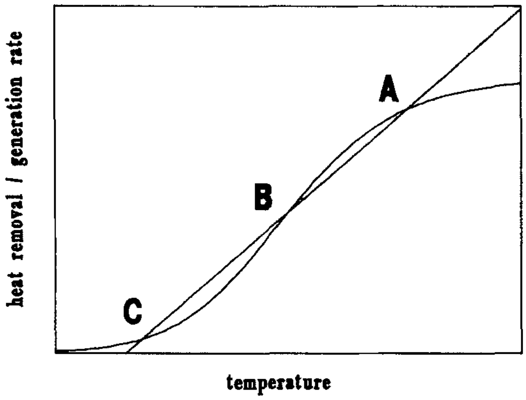

Figure 1. Nonisothermal CSTR steady-state diagram.

the definition of stability. As a result, stability regions for the operator N can be identified.

To present a more demanding notion of stability that guarantees finite amplification of input signals we introduce the concepts of *gain over set* and *incremental gain over set.*

Definition. The *dynamic gain* of the unbiased operator $N: L_{pe}^m \longrightarrow L_{pe}^n$ *over the set* $W \subseteq L_p^m$ (or simply set gain) is defined as

$$\|N\|_{pW} \triangleq \sup_{\substack{u \in W \\ u \neq 0}} \frac{\|Nu\|_p}{\|u\|_p} \tag{20}$$

Definition. The *dynamic incremental gain* of the unbiased operator $N: L_{pe}^m \longrightarrow L_{pe}^n$ *over* the set $W \subset L_p^m$ is defined as

$$\|N\|_{\Delta pW} \triangleq \sup_{\substack{u_1, u_2 \in W \\ u_1 \neq u_2}} \frac{\|Nu_1 - Nu_2\|}{\|u_1 - u_2\|_p} \tag{21}$$

Definition. A system $N: L_{pe}^m \longrightarrow L_{pe}^n$ is *finite gain stable over the set* W if $\|N\|_{pW} < \infty$.

Definition. A system $N: L_{pe}^m \longrightarrow L_{pe}^n$ is *finite incremental gain stable over the set* W if $\|N\|_{\Delta pW} < \infty$.

The set W identifies these input signals that are physically important and is often selected as

$$W = \{u: R \longrightarrow R^m: t \longrightarrow u(t) \quad \text{where} \quad \|u\|_p < \infty, u_{\min} \leq u(t) \leq u_{\max} \; \forall \, t \geq 0\}$$

Computation of gains over sets

For the above definitions to become operational and useful it is necessary that the gains $\|N\|_{pW}$ and $\|N\|_{\Delta pW}$ be computable. Next we present a series of theorems that make feasible the computation of these gains.

Theorem 1.

Let $N: L_{pe}^m \longrightarrow L_{pe}^n: u \longrightarrow y: 0 \longrightarrow 0$ with $W = \{u \in L_p^m$ such that $0 < \|u\|_p \leq \delta\}$. Then

$$\|N\|_{pW} \triangleq \sup_{u \in W} \frac{\|Nu\|_p}{\|u\|_p} = \sup_{0 < \delta' \leq \delta} \frac{\sup_{\|u\|_p = \delta'} \|Nu\|_p}{\delta'} \tag{22}$$

For proof, see Appendix B.

Theorem 2. Let $N: L_{pe}^m \longrightarrow L_{pe}^n$ and $L_{u_o}: L_{pe}^m \longrightarrow L_{pe}^n$ be an unbiased operator and its linearization around the trajectory u_o, respectively. Let also W be a *convex* subset of L_p^m. Then, for $p \in [1, \infty]$,

$$\|N\|_{\Delta pW} \hat{=} \sup_{\substack{u_1,u_2 \in W \\ u_1 \neq u_2}} \frac{\|Nu_1 - Nu_2\|_p}{\|u_1 - u_2\|_p} = \sup_{u_o \in W} \|L_{u_o}\|_p \qquad (23)$$

For proof, see Appendix D.

Lemma. Let N: $L_{pe}^m \rightarrow L_{pe}^n$: $0 \rightarrow 0$: $u \rightarrow y$ be realized by the equations

$$\dot{x}(t) = \phi(x(t), u(t)), \quad x(0) = a, \quad \phi(a, 0) = 0$$
$$y(t) = Cx(t) + d, \quad Ca + d = 0 \qquad (24)$$

with $x(t) \in R^\ell$, $u(t) \in R^m$, $y(t) \in R^n$, and ϕ: $R^\ell \times R^m \rightarrow R^\ell$ locally Lipschitz with respect to its first argument. Then the linearization L_{u_o}: $L_{pe}^m \rightarrow L_{pe}^n$ of N about $u_o(\cdot)$ and $x_o(\cdot)$ is realized by

$$\Delta\dot{x}(t) = A(t)\Delta x(t) + B(t)u(t), \quad \Delta x(0) = 0$$
$$y(t) = C\Delta x(t) \qquad (25)$$

where

$$\dot{x}_o(t) = \phi(x_o(t), u_o(t)) \quad x_o(0) = a$$
$$A(t) = \frac{\partial\phi}{\partial x}(x_o(t), u_o(t))$$
$$B(t) = \frac{\partial\phi}{\partial u}(x_o(t), u_o(t))$$

For proof, see Willems (1971).

Lemma. Let the linear operator X: $L_{\infty e}^m \rightarrow L_{\infty e}^n$: $u \rightarrow y$ be defined by

$$(Xu)(t) = \int_0^t W(t, \tau)u(\tau)\, d\tau$$

where $W(t, \tau) \in R^{n\times m}$. Then

$$\|X\|_\infty \hat{=} \sup_{\|u\|_\infty = 1} \|Xu\|_\infty = \sup_{t\in[0,\infty)} \int_0^t \|W(t, \tau)\|_i\, d\tau \qquad (26)$$

where $\|W(t, \tau)\|_i$ denotes *any* induced norm of the matrix $W(t, \tau)$.

For proof, see Desoer and Vidyasagar (1975).

Theorem 3. The *dynamic set-incremental gain* of the operator N: $L_{\infty e}^m \rightarrow L_{\infty e}^n$ defined by Eq. 24 is given as the solution of the following *nonsmooth* optimal control problem

$$\|N\|_{\Delta\infty W} = \sup_{t\in[0,\infty)} \left[- \inf_{u_o \in W} \int_0^t - \|C\Phi(t, \tau)B(\tau)\|_i\, d\tau \right]$$

where

$$\Phi: [0, \infty) \rightarrow R^{\ell\times\ell}$$
$$\dot{\Phi}(\tau) = -\Phi(\tau)A(\tau), \quad \Phi(t) = I$$
$$\dot{x}_o(\tau) = \phi(x_o(\tau), u_o(\tau)) \quad x_o(0) = a$$
$$A(\tau) = \frac{\partial\phi}{\partial x}(x_o(\tau), u_o(\tau))$$
$$B(\tau) = \frac{\partial\phi}{\partial u}(x_o(\tau), u_o(\tau))$$

For proof, see Appendix E.

Remarks.

- If $W = L_p^m$, then the above theorem yields the *incremental gain* of N.
- If the inf in the above theorem can be substituted by min, then the nonsmooth optimal control problem is solvable using the nonsmooth minimum principle.

Example 3. For example 1, described earlier, consider the heater modeled by Eqs. 18 with $u \hat{=} \Delta F$ and $y \hat{=} \Delta T$. We shall compute $\|N\|_{\infty W}$ for $W \hat{=} \{u \in L_\infty | -F_s \leq u(t) \leq F_s, t \in [0, \infty)\}$ and $\|N\|_{\Delta\infty W}$ for $W \hat{=} \{u: R \rightarrow R | -\epsilon \leq u(t) \leq \delta, t \in [0, \infty)\}$. To compute $\|N\|_{\infty W}$, we get, from Eq. C15

$$\sup_{\substack{t\geq 0 \\ |\Delta F(t)| \leq \Delta F_{max}}} |T(t) - T_s| = T(\infty, -\Delta F_{max}) - T_s$$

Then, theorem 1 suggests that

$$\|N\|_{\infty W} = \sup_{0<\Delta F_{max}\leq F_s} \frac{|\Delta T(\infty, -\Delta F_{max})|}{\Delta F_{max}}$$
$$\stackrel{(C17)}{=} \sup_{0<\Delta F_{max}\leq F_s} \frac{|T_i - T_s|}{\left| F_s - \Delta F_{max} + \dfrac{UA}{\rho c_p} \right|} = (T_s - T_i)\frac{\rho c_p}{UA} < \infty \qquad (27)$$

To compute $\|N\|_{\Delta\infty W}$, we set up the following nonsmooth optimal control problem, as suggested by theorem 3:

$$\sup_{t\in[0,\infty)} \left[- \min_{\Delta F\in W} \int_0^t - \left| \Phi(\tau)\frac{1}{V}\left(\Delta T(\tau) + \frac{UA(T_c - T_i)}{\rho c_p F_s + UA}\right) \right| d\tau \right]$$

under the constraints

$$\frac{d\Phi(\tau)}{d\tau} = \Phi(\tau)\frac{1}{V}(\Delta F(\tau) + \frac{UA}{\rho c_p} + F_s), \quad \Phi(t) = 1$$
$$\frac{d\Delta T(\tau)}{d\tau} = -\frac{1}{V}\left[\Delta F(\tau)\Delta T(\tau) + \left(\frac{UA}{\rho c_p} + F_s\right)\Delta T(\tau) + \frac{UA(T_c - T_i)}{\rho c_p F_s + UA}\Delta F(\tau)\right], \quad \Delta T(0) = 0 \qquad (28)$$

To characterize the optimum we solve a number of fixed terminal time nonsmooth optimal control problems. For each of these problems the Hamiltonian, from Eq. 10, is:

$$H(\Delta F(\tau)) \hat{=} H(\Phi(\tau), \Delta T(\tau), p_1(\tau), p_2(\tau), \Delta F(\tau))$$
$$= -\lambda \left| \Phi(\tau)\frac{1}{V}\left(\Delta T(\tau) + \frac{UA(T_c - T_i)}{\rho c_p F_s + UA}\right) \right|$$
$$+ p_1(\tau)\Phi(\tau)\frac{1}{V}(\Delta F(\tau) + \frac{UA}{\rho c_p} + F_s) - p_2(\tau)$$
$$\cdot \frac{1}{V}\left[\Delta F(\tau)\Delta T(\tau) + \left(\frac{UA}{\rho c_p} + F_s\right)\Delta T(\tau) + \frac{UA(T_c - T_i)}{\rho c_p F_s + UA}\Delta F(\tau)\right] \qquad (29)$$

The adjoint "equations," from Eq. 12, are:

$$-\dot{p}_1(\tau) \in \partial_\Phi H = co\left\{\lim \frac{\partial H}{\partial \Phi_i} \quad \text{where} \quad \lim_i \Phi_i = \Phi(\tau), \Phi_i \neq 0\right\}$$
$$= \{-a(\tau) + p_1(\tau)b(\tau)\}, \quad \text{if } \Phi(\tau) > 0 \quad (30)$$

$$= \{a(\tau) + p_1(\tau)b(\tau)\}, \quad \text{if } \Phi(\tau) < 0 \qquad (31)$$

$$= [-a(\tau) + p_1(\tau)b(\tau), a(\tau) + p_1(\tau)b(\tau)], \quad \text{if } \Phi(\tau) = 0 \qquad (32)$$

and

$$-\dot{p}_2(\tau) \in \partial_{\Delta T} H = co\left\{\lim \frac{\partial H}{\partial \Delta T_i} \quad \text{where} \quad \lim_i \Delta T_i = \Delta T(\tau), \Delta T_i \neq -\frac{UA(T_c - T_i)}{\rho c_p F_s + UA}\right\}$$

$$= \{-c(\tau) + p_2(\tau)d(\tau)\}, \quad \text{if } \Delta T(\tau) + \frac{UA(T_c - T_i)}{\rho c_p F_s + UA} > 0 \qquad (33)$$

$$= \{c(\tau) + p_2(\tau)d(\tau)\}, \quad \text{if } \Delta T(\tau) + \frac{UA(T_c - T_i)}{\rho c_p F_s + UA} < 0 \qquad (34)$$

$$= [-c(\tau) + p_2(\tau), c(\tau) + p_2(\tau)d(\tau)], \quad \text{if } \Delta T(\tau) + \frac{UA(T_c - T_i)}{\rho c_p F_s + UA} = 0 \qquad (35)$$

The transversality conditions, from Eqs. 13 and A4, are:

$$p_1(0) = p_2(t) = 0 \qquad (36)$$

where

$$a(\tau) \doteq \lambda \left| \frac{1}{V}\left(\Delta T(\tau) + \frac{UA(T_c - T_i)}{\rho c_p F_s + UA}\right) \right|$$

$$b(\tau) \doteq \frac{1}{V}\left(\Delta F(\tau) + \frac{UA}{\rho c_p} + F_s\right)$$

$$c(\tau) \doteq \lambda \left| \Phi(\tau) \frac{1}{V} \right|$$

$$d(\tau) \doteq -\frac{1}{V}\left(\Delta F(\tau) + \frac{UA}{\rho c_p} + F_s\right)$$

The solution of this problem, presented in Appendix F, yields

$$\|N\|_{\Delta\infty W} = \frac{(T_c - T_i)\dfrac{UA}{\rho c_p}}{\left[\dfrac{UA}{\rho c_p} + F_s - \epsilon\right]^2} \qquad (37)$$

Remarks.

- Equations 32 and 35 indicate that for $\Phi(\tau) = 0$ and $[UA(T_c - T_i)]/(\rho c_p F_s + UA) = 0$, the values of $\dot{p}_1(\tau)$ and $\dot{p}_2(\tau)$ are not uniquely determined. Instead, closed intervals where they lie are known. This, however, has no impact on $p_1(\tau)$ and $p_2(\tau)$, provided that Eqs. 32 and 35 hold only on sets of measure zero, namely at isolated instances. If this is not the case, the optimal control problem posed above can be viewed as a free-terminal-time problem and the additional condition

$$H(x(\tau), p(\tau), u(\tau)) = 0 \quad \forall \tau \in (0, t]$$

can be invoked.

- It should be noted that the upper bound for $\Delta F(\tau)$ is not involved in the preceding computations. Therefore for

$$W \doteq \{u: \mathbf{R} \rightarrow \mathbf{R} \mid -\epsilon \leq u(t) < \infty, t \in [0, \infty)\}$$

we get the same set-incremental gain. In particular, if $W \doteq \{u: \mathbf{R} \rightarrow \mathbf{R} \mid -F_s \leq u(t) < \infty, t \in [0, \infty)\}$, then $\|N\|_{\Delta\infty W} = (T_c - T_i)\rho c_p/UA$. This corresponds to ***all physical signals*** that may enter the system and clearly exemplifies the significance of working within subsets of L_p.

- It can be verified that if W contains 0, then

$$\|N\|_{\Delta pW} \geq \|N\|_{pW}$$

(See Appendix G.) Therefore, as Eqs. 37 (for $\epsilon = F_s$) and 20 suggest, $(T_c - T_i)\rho c_p/AU \geq (T_s - T_i)\rho c_p/AU$ should hold. Indeed this is the case, since $T_c \geq T_s$.

- For this particular example the equations involved are explicitly solvable. In general, two-point boundary value problems will arise which require numerical solution, as demonstrated next.

Example 4. Consider the CSTR of example 2, with parameter values as in Table 1. To compute $\|N\|_{\Delta\infty W}$, we set up the following nonsmooth optimal control problem:

Find

$$\sup_{t \geq 0}\left[-\min_{\Delta Q \in W_\beta} \frac{-1}{\rho c_p V}\int_0^t |\Phi_2(\tau)|\, d\tau\right]$$

under the dynamic constraints

$$\frac{dC_A(\tau)}{d\tau} = \frac{F}{V}(C_{Ai} - C_A(\tau)) - C_A(\tau)\alpha e^{-E/RT(\tau)} \doteq \phi_1(x(\tau), u(\tau)), \quad C_A(0) = C_{As}$$

$$\frac{dT(\tau)}{d\tau} = \frac{F}{V}(T_i - T(\tau)) - \frac{\Delta H_R}{c_p \rho} C_A(\tau)\alpha e^{-E/RT(\tau)} - \frac{Q_s + \Delta Q(\tau)}{\rho c_p V} \doteq \phi_2(x(\tau), u(\tau)), \quad T(0) = T_s$$

$$\frac{d\Phi_1(\tau)}{d\tau} = \Phi_1(\tau)\left(\frac{F}{V} + \alpha e^{-E/RT(\tau)}\right) + \Phi_2(\tau)\frac{\Delta H_R}{c_p \rho}\alpha e^{-E/RT(\tau)} \doteq \phi_3(x(\tau), u(\tau)), \quad \Phi_1(t) = 0$$

$$\frac{d\Phi_2(\tau)}{d\tau} = \Phi_1(\tau) C_A(\tau)\alpha \frac{E}{RT^2(\tau)} e^{-E/RT(\tau)} + \Phi_2(\tau)\left(\frac{F}{V} + \frac{\Delta H_R}{c_p \rho} C_A(\tau)\alpha \frac{E}{RT^2(\tau)} e^{-E/RT(\tau)}\right) \doteq \phi_4(x(\tau), u(\tau)), \quad \Phi_2(t) = 1$$

Table 1. Parameters Used for Example 4

$F = 1.133\ m^3/h$	$\Delta H_R = -69{,}775$ J/mol
$V = 1.36\ m^3$	$c_p = 3{,}140$ J/kg · K
$C_{Ai} = 8{,}008\ mol/m^3$	$\rho = 800.8\ kg/m^3$
$\alpha = 7.08 \times 10^7\ h^{-1}$	Steady State
$E/R = 8{,}375$ K	$Q_s = 1.055 \times 10^8$ J/h
$T_i = 373.3$ K	$T_s = 547.6$ K
	$C_{As} = 393.2\ mol/m^3$

where $W_\beta \hat{=} \{\Delta Q: \boldsymbol{R} \rightarrow \boldsymbol{R} | -\beta Q_s \geq \Delta Q(\tau) \leq \beta Q_s \ \forall\ \tau \in [0, t]\}$. The Hamiltonian, according to Eq. 10, is

$$H(x(\tau), p(\tau), u(\tau)) = -\lambda \frac{1}{\rho c_p V} |\Phi_2(\tau)| + \sum_{i=1}^{4} p_i(\tau)\phi_i(x(\tau), u(\tau))$$

The minimum principle then yields

$$H(x(\tau), p(\tau), u(\tau)) = \min \{-p_2(\tau) \frac{1}{\rho c_p V} w + E(x(\tau), p(\tau)) | -\beta Q_s \leq w \leq \beta Q_s\}$$

which in turn dictates that $\Delta Q(\tau) = \beta Q_s \cdot sgn[p_2(\tau)]$ when $p_2(\tau) \neq 0$. The adjoint "equations" are set up according to Eq. 12 and the transversality conditions become $p_1(t) = p_2(t) = p_3(0) = p_4(0) = 0$. For $\lambda = 0$, the adjoint equations accept only the solution $p_1(\tau) = p_2(\tau) = p_3(\tau) = p_4(\tau) = 0$ (the problem is normal).

For $\lambda = 1$, we first presume a profile for ΔQ and a sign for Φ_2. We then solve numerically the resulting two point boundary value problem through forward and backward integration with corrections on the presumed profile of ΔQ. For $\pm 5\%$ and $\pm 10\%$ maximum deviation of Q from its steady state value ($\beta = 0.05$, $\beta = 0.10$), and for fixed terminal time, the control ΔQ which yields the extremum is $\Delta Q = \beta Q_s$. By repeating the solution for increasing terminal time we observe that the objective function levels off at 5.15×10^{-7} and 5.31×10^{-7} K/J/h as $t \rightarrow \infty$. These are the L_∞ incremental gains of the CSTR over the sets $W_{0.05}$ and $W_{0.10}$, respectively, Figure 2. It should be stressed that this CSTR is not expected to be incrementally stable over *all* physically meaningful input signal sets, due to the presence of multiple steady states.

Indeed, an infinitesimal change in ΔQ near the middle operating point B, Figure 1, will result in a finite change in the steady-state temperature, thus resulting in infinite incremental gain.

Remark. By applying Eq. 25 to the linearized model of the CSTR around steady state A it is straightforward to calculate the incremental gain of the corresponding linear operator (which coincides with its ∞ norm). Its value is 4.94×10^{-7} K/J/h, relatively close and, as expected, *below* the values of the set-incremental gains computed above.

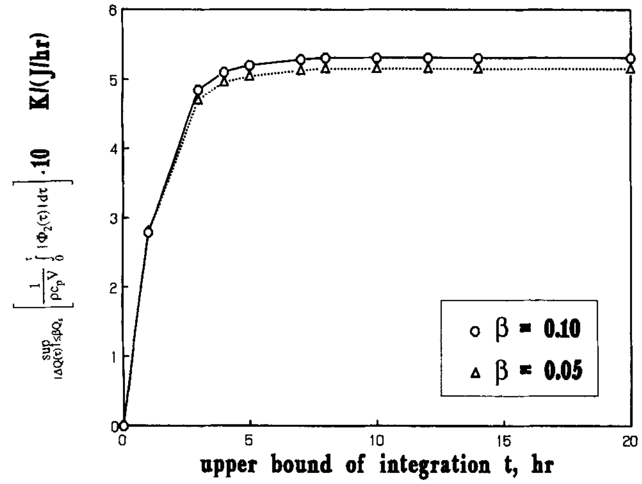

Figure 2. L_∞-set-incremental gain of CSTR.

Input-Output Performance

The performance issue for a closed loop system refers to the deviation of this system's output from a desired value. There are two distinct formulations of the performance issue, namely the set-point tracking and disturbance rejection problems. Without loss of generality we consider next the disturbance rejection problem. For the classical feedback scheme, Figure 3, this problem is mathematically quantified by the requirement that the gain of the nonlinear operator $S(N, K)$: $d \rightarrow y$ be small. Since N and/or K are nonlinear, it is clear that gains over sets must be considered. In the example that follows we demonstrate the use of the L_∞ set gain of the operator $S(N, K)(\|S(N, K)\|_{\infty W})$ as a closed-loop performance analysis tool.

Example 5. Consider the heater modeled by Eq. 18, with manipulated input $u \hat{=} \Delta F/F_s$, output $h \hat{=} \Delta T/T_s$, and disturbance $d \hat{=} \Delta T_i/T_{is}$. A model reference scheme controller, which happens to be a PI controller $K(s) = (s + a)/b\epsilon s$ (Table 2), is used to regulate the system. The closed-loop operator S: $d \rightarrow y$ is realized by the following equations:

$$\dot{y}(t) = \frac{1}{V}\left[F_s\left(1 + \frac{1}{b\epsilon}(x_K(t) - y(t))\right) \cdot \frac{T_{is}(1 + d(t)) - T_s(1 + y(t))}{T_s} + \frac{UA(T_c - T_s(1 + y(t)))}{\rho c_p T_s}\right]$$

$$\dot{x}_k(t) = -ay(t)$$

$$y(0) = x_k(0) = 0$$

If $W \hat{=} \{d \in L_\infty | -\delta \leq d(t) \leq \delta\}$ then, by theorem 1,

$$\|S\|_{\infty W} = \sup_{d \in W} \frac{\|y\|_\infty}{\|d\|_\infty} = \sup_{0 < \delta' \leq \delta} \frac{\sup_{t, \|d\|_\infty = \delta'} |y(t)|}{\delta'}$$

The minimum principle yields

$$H(x(\tau), p(\tau), d(\tau)) = \min \left\{p(\tau) \frac{F_s}{V}\left(+ \frac{V(x_K(\tau) - y(\tau))}{F_s(T_{is}/T_s - 1)\epsilon}\right) \frac{T_{is}}{T_s} \cdot w + E(x(\tau), p(\tau)): \delta' \leq w \leq \delta'\right\}$$

which in turn dictates that

$$d(\tau) = -\delta' sgn\left[p(\tau)\left(1 + \frac{V(x_K(\tau) - y(\tau))}{F_s(T_{is}/T_s - 1)\epsilon}\right)\right]$$

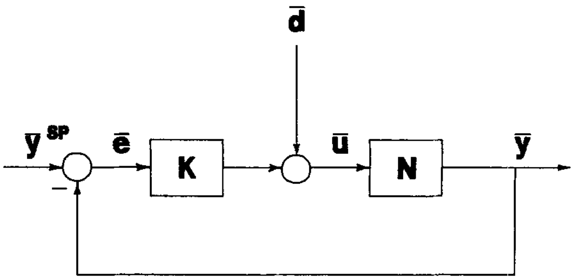

Figure 3. Classical feedback scheme.

Table 2. Parameters Used for Example 5

$V = 1\ m^3$	$a = \left(F_s + \dfrac{UA}{\rho c_p}\right) \Big/ V$
$T_{is} = 300\ K$	
$T_c = 375\ K$	$b = \dfrac{F_s}{V}\left(\dfrac{T_{is}}{T_s} - 1\right)$
$\dfrac{UA}{\rho c_p} = 28\ m^3/h$	
Steady State	$c = \dfrac{F_s T_{is}}{V T_s}$
$F_s = 2\ m^3/h$	
$T_s = 370\ K$	$\epsilon = 0.03\ h$

The adjoint "equations" are set up according to Eq. 12 and the transversality conditions become

$$p_1(t) = -1, \quad \text{if } y(t) > 0$$
$$= +1, \quad \text{if } y(t) < 0$$
$$\in [-1, 1], \quad \text{if } y(t) = 0$$
$$p_2(t) = 0$$

To solve the nonsmooth optimal control problem, we employ the following iteration scheme.

1. A terminal time t is assumed.
2. A sign is chosen for the coefficient of the disturbance $d(\tau)$ in the Hamiltonian.
3. Application of minimum principle yields a candidate $d(\tau)$ which is used to integrate the state equations forward, and then the costate equations backward.
4. The sign of the coefficient of $d(\tau)$ in the Hamiltonian, and the value of the Hamiltonian, are evaluated for the states and costates identified in step 3.
5. If the sign in 4 does not coincide with that in 2, then step 2 is repeated.
6. If the value of the Hamiltonian is not zero, a new terminal time t is chosen and step 1 is repeated. The results are depicted in Figure 4.

Remarks

- The extremizing d is bang-bang with one switching from $-\delta$ to δ at T.
- The nonlinear operator's set gain tends to the linearized system's gain as the maximum allowable value of the disturbance signal tends to zero.
- The nonlinear operator's set gain is strongly dependent on the maximum allowable value of the disturbance. As a result, a 5% deviation of the disturbance from its steady state value causes 3.8% deviation in the output of the linearized system while the output deviation of the nonlinear system is 4.9% which is 30% larger than that predicted by linear analysis.
- The time after the switching at which the extremum of $|y(t)|$ is achieved is a monotonically increasing function of δ. This indicates that the nonlinear system's behavior becomes increasingly sluggish as the disturbance magnitude increases.

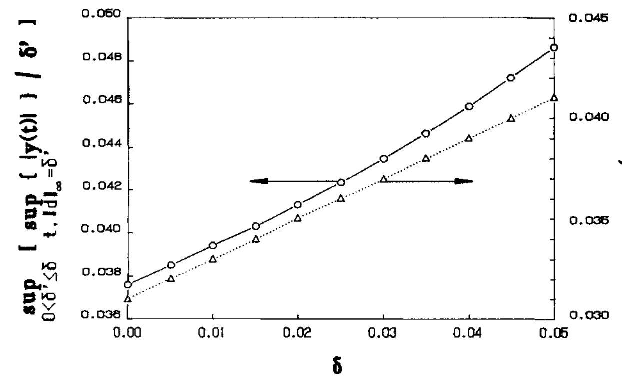

Figure 4. Closed loop L_∞-set-gain of heater.

Acknowledgment

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Appendix A: Mathematical Background on Nonsmooth Calculus

Definition. The function f: $R^n \rightarrow R$ is *locally Lipschitz*, of rank κ, at $x \in R^n$ if there exists $\kappa > 0$ such that

$$|f(y_1) - f(y_2)| \leq \kappa \|y_1 - y_2\|$$

for y_1, y_2 in a neighborhood of x.

Definition. Let f: $R^n \rightarrow R$ be locally Lipschitz of rank κ at x. Then the *generalized directional derivative* of f at x in the direction v, denoted $f^\circ(x; v)$ is defined as follows:

$$f^\circ(x; v) \triangleq \limsup_{\substack{y \to x \\ t > 0 \\ t \to 0}} \frac{f(y + tv) - f(y)}{t}$$

$$= \lim_{\epsilon \to 0} \sup_{\|y - x\| < \epsilon\delta} \sup_{0 < t < \epsilon} \frac{f(y + tv) - f(y)}{t} \quad \text{(A1)}$$

Remark. Observe that the point x where the generalized directional derivative is considered is only implicitly introduced into the above ratios. If f is differentiable at x, *then Eq. A1* reduces to

$$f^\circ(x; v) = \lim_{t \to 0} \frac{f(x + tv) - f(x)}{t}$$

which is the directional derivative of f in the direction v.

Definition. The *generalized gradient* of f at x, denoted $\partial f(x)$, is defined as the subset of R^n given by

$$\partial f(x) = \{\zeta \in R^n : f^\circ(x; v) \geq \langle \zeta, v \rangle \ \forall\, v \in R^n\} \quad \text{(A2)}$$

Remarks.

- Observe that $\partial f(x)$ is not a single point in R^n but a whole set. In fact, it can be shown that $\partial f(x)$ is a nonempty, convex, closed and bounded subset of R^n. Moreover, knowing $f^\circ(x; v)$ is equivalent to knowing $\partial f(x)$ since, by Eq. A2

$$f^\circ(x; v) = \max \{\langle \zeta, v \rangle : \zeta \in \partial f(x)\}$$

When f is differentiable everywhere, $f^\circ(x; v)$ reduces to the gradient of f.

- The computation of $\partial f(x)$ from its definition is hardly ever carried out in practice. Instead, just as in differential calculus, a subsequent fact is invoked which covers the most significant class of nondifferentiable functions occurring in practice,

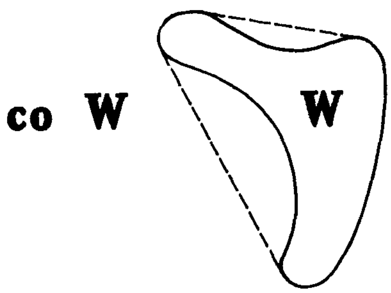

Figure 5. The convex hull of a set in R^2.

namely those which fail to be differentiable only at isolated points.

Definition. A set W is *convex* if

$$\alpha x + (1 - \alpha)\tilde{x} \in W \; \forall \; x, \tilde{x} \in W, \forall \alpha \in [0, 1]$$

Definition. The *convex hull* of a set W, denoted coW, is the smallest convex set containing W. (See also Figure 5.)

Fact (Clarke, 1983). Let $f: \boldsymbol{R}^n \rightarrow \boldsymbol{R}$ be Lipschitz near x. Let Ω_f denote the set of points around x where f fails to be differentiable and S any other set of isolated points. Then

$$\partial f(x) = co\{r \mid r = \lim \nabla f(x_i) \mid x_i \rightarrow x, x_i \notin (\Omega_f \cup S)\} \quad \text{(A3)}$$

Remark. The meaning of Eq. A3 is the following: consider any sequence x_i (hence *not uniquely* determined) in $\boldsymbol{R}^n$ converging to x while avoiding both S and Ω_f and such that the sequence $\nabla f(x_i)$ converges; then the convex hull of all such limit points is $\partial f(x)$.

Definition. Let $C \subset \boldsymbol{R}^n$. We define the *distance function* d_c: $\boldsymbol{R}^n \rightarrow \boldsymbol{R}$ by

$$d_c(x) \triangleq \inf_{c \in C} \|x - c\|$$

Definition. The *tangent cone* $T_c(x)$ to $C \subset \boldsymbol{R}^n$ at $x \in C$ is defined as the set

$$T_c(x) \triangleq \{v \in \boldsymbol{R}^n: d_c^o(x; v) = 0\}$$

Definition. The *normal cone* $N_c(x)$ to $C \subset \boldsymbol{R}^n$ at $x \in C$ is defined as:

$$N_c(x) \triangleq \{\zeta \in \boldsymbol{R}^n: \langle \zeta, v \rangle \leq 0, \forall \; v \in T_c(x)\}$$

Remark. It can be shown that $N_{\boldsymbol{R}^n}(x) = T_{\boldsymbol{R}^n}(x) = \{0\}$ and $N_a(x) = T_a(x) = \boldsymbol{R}^n$ for $a \in \boldsymbol{R}^n$. It also holds that if $x \in C \subset \boldsymbol{R}^n$ and $x = (x_1, x_2)$ with $x_1 \in C_1$ and $x_2 \in C_2$ then

$$T_{c_1 \times c_2}(x) = T_{c_1}(x_1) \times T_{c_2}(x_2) \quad \text{and}$$
$$N_{c_1 \times c_2}(x) = N_{c_1}(x_1) \times N_{c_2}(x_2) \quad \text{(A4)}$$

Appendix B: Computation of $\|N\|_{pW}$

Theorem 1. Let $N: L_{pe}^m \rightarrow L_{pe}^n: u \rightarrow y: 0 \rightarrow 0$ with $W \triangleq \{u: \boldsymbol{R} \rightarrow \boldsymbol{R}^m$ such that $0 < \|u\|_p \leq \delta\}$. Then

$$\|N\|_{pW} \triangleq \sup_{u \in W} \frac{\|Nu\|_p}{\|u\|_p} = \sup_{0 < \delta' \leq \delta} \frac{\sup_{\|u\|_p = \delta'} \|Nu\|_p}{\delta'}$$

Proof. It holds that

$$\sup_{0<\|u\|_p \leq \delta} \frac{\|Nu\|_p}{\|u\|_p} \geq \sup_{\|u\|_p = \delta'} \frac{\|Nu\|_p}{\|u\|_p} = \sup_{\|u\|_p = \delta'} \frac{\|Nu\|_p}{\delta'}$$
$$= \frac{\sup_{\|u\|_p = \delta'} \|Nu\|_p}{\delta'} \; \forall \; \delta' \in (0, \delta] \quad \text{(B1)}$$

$$\Longrightarrow \sup_{0<\|u\|_p \leq \delta} \frac{\|Nu\|_p}{\|u\|_p} \geq \sup_{0<\delta' \leq \delta} \frac{\sup_{0<\|u\|_p = \delta'} \|Nu\|_p}{\delta'} \quad \text{(B2)}$$

Now let

$$\sup_{0<\|u\|_p = \delta'} \frac{\|Nu\|_p}{\|u\|_p} = \lambda \quad \text{(B3)}$$

Then $\forall \; \epsilon > 0 \; \exists \; u_\epsilon$ with $0 < \|u_\epsilon\|_p = \delta' \leq \delta$ such that

$$\frac{\|Nu_\epsilon\|_p}{\|u_\epsilon\|_p} \geq \lambda - \epsilon \Longrightarrow \frac{\|Nu_\epsilon\|_p}{\delta'} \geq \lambda - \epsilon \Longrightarrow \frac{\sup_{0<\|u_\epsilon\|_p = \delta'} \|Nu_\epsilon\|_p}{\delta'} \geq \lambda - \epsilon$$

$$\Longrightarrow \sup_{0<\delta' \leq \delta} \frac{\sup_{0<\|u_\epsilon\|_p = \delta'} \|Nu_\epsilon\|_p}{\delta'} \geq \sup_{0<\|u\|_p \leq \delta} \frac{\|Nu\|}{\|u\|_p} - \epsilon \; \forall \; \epsilon > 0 \quad \text{(B4)}$$

Since Eq. B4 holds for any ϵ and Eq. B2 holds simultaneously, it must be

$$\sup_{0<\|u\|_p \leq \delta} \frac{\|Nu\|_p}{\|u\|_p} = \sup_{0<\delta' \leq \delta} \frac{\sup_{0<\|u\|_p = \delta'} \|Nu\|_p}{\delta'} \quad \text{QED} \quad \text{(B5)}$$

Appendix C: L_∞-Gain of a Heater

It holds that

$$\|N\|_\infty = \sup_{\Delta F \in L_\infty} \frac{\sup_{t \geq 0} |\Delta T(t)|}{\sup_{t \geq 0} |\Delta F(t)|}.$$

Based on the theorem of Appendix B we have

$$\|N\|_\infty = \sup_{\Delta F_{\max} > 0} \frac{\sup_{\substack{\|\Delta F\|_\infty = \Delta F_{\max} \\ t > 0}} |\Delta T(t)|}{\Delta F_{\max}}.$$

Therefore, we need to first identify

$$\sup_{\substack{t \geq 0 \\ |\Delta F(t)| \leq \Delta F_{\max}}} |\Delta T(t)|$$

given that

$$T(t) = \frac{1}{V} \int_0^t \left[F(t)(T_i - T(\tau)) + \frac{UA(T_c - T(\tau))}{\rho c_p} \right] d\tau.$$

Since

$$\sup_{t, \Delta F} |\Delta T(t)|$$
$$= \max \left\{ \sup_{t, \Delta F} (T_{\max}(t) - T_s), \sup_{t, \Delta F} (T_s - T_{\min}(t)) \right\} \quad \text{(C1)}$$

we need identify only $T_{min}(t)$ and $T_{max}(t)$ for all fixed t in $[0, \infty)$. The Hamiltonian of the corresponding optimal control problems can be written as

$$H(\tau) = (p(\tau) + \xi)\frac{1}{V}\left[F(\tau)(T_i - T(\tau)) + \frac{UA(T_c - T(\tau))}{\rho c_p}\right] \quad (C2)$$

with $\xi = 1$ or $\xi = -1$ depending on whether $T_{min}(t)$ or $T_{max}(t)$ is sought, respectively. Thus

$$\begin{aligned} F(\tau) &= F_{min} \quad \text{if} \quad (p(\tau) + \xi)(T_i - T(\tau)) > 0 \\ F(\tau) &= F_{max} \quad \text{if} \quad (p(\tau) + \xi)(T_i - T(\tau)) < 0 \end{aligned} \quad (C3)$$

The costate equation implies that

$$\left.\begin{aligned} \dot{p}(\tau) &= -\frac{\partial H}{\partial x} = (p(\tau) + \xi)\frac{1}{V}\left(F(\tau) + \frac{UA}{\rho c_p}\right) \\ p(t) &= 0 \end{aligned}\right\}$$

$$\Longrightarrow p(\tau) + \xi = \xi \exp\left[\int_t^\tau \frac{1}{V}\left(F(r) + \frac{UA}{\rho c_p}\right) dr\right] \quad (C4)$$

Therefore

$$sgn(p(\tau) + \xi) = sgn\xi \quad (C5)$$

and since, for physical reasons, it is always

$$T_i - T(\tau) < 0 \quad (C6)$$

we conclude that

$$sgn[(p(\tau) + \xi)(T_i - T(\tau))] = -sgn\,\xi. \quad (C7)$$

The resulting extremal policies are

$$F(\tau) = F_{min}\,(\Delta F(\tau) = -\Delta F_{max}), \tau \in (0, t), \quad (C8)$$

to achieve $T_{max}(t)$ and

$$F(\tau) = F_{max} \quad (\Delta F(\tau) = \Delta F_{max}), \tau \in (0, t), \quad (C9)$$

to achieve $T_{min}(t)$. The extremal value of $T(t)$ is

$$T(t) = \exp\left[-\frac{1}{V}\left(F + \frac{UA}{\rho c_p}\right)t\right]\left(T_i + \frac{UA}{\rho c_p F_s + UA}(T_c - T_i)\right) + \frac{FT_i + \dfrac{UAT_c}{\rho c_p}}{F + \dfrac{UA}{\rho c_p}}\left(1 - \exp\left[-\frac{1}{V}\left(F + \frac{UA}{\rho c_p}\right)t\right]\right) \quad (C10)$$

where F is either F_{min} or F_{max}.

From Eq. C10 it is straightforward to show that for $t_1 > t_2$

$$T(t_1) > T(t_2) \quad \text{when } F = F_{min} \quad (C11)$$

and

$$T(t_1) < T(t_2) \quad \text{when } F = F_{max} \quad (C12)$$

Thus if $\Delta T(t, \Delta F)$ denotes the solution of Eq. C1 at time t corresponding to the policy ΔF, we have that

$$\sup_{\substack{t \geq 0 \\ |\Delta F(t)| \leq \Delta F_{max}}} T(t) = \sup_{t \geq 0} T_{max}(t) = T(\infty, F_{min} - F_s) = T_i + \frac{UA}{\rho c_p F_{min} + UA}(T_c - T_i) \quad (C13)$$

$$\inf_{\substack{t \geq 0 \\ |\Delta F(t)| \leq \Delta F_{max}}} T(t) = \inf_{t \geq 0} T_{min}(t) = T(\infty, F_{max} - F_s) = T_i + \frac{UA}{\rho c_p F_{max} + UA}(T_c - T_i) \quad (C14)$$

Given that $F_{min} = F_s - \Delta F_{max}$ and $F_{max} = F_s + \Delta F_{max}$ we have

$$\sup_{\substack{t \geq 0 \\ |\Delta F(t)| \leq \Delta F_{max}}} |T(t) - T_s| = \max\{T(\infty, -\Delta F_{max}) - T_s, T_s - T(\infty, \Delta F_{max})\} = T(\infty, -\Delta F_{max}) - T_s. \quad (C15)$$

Therefore, the L_∞-gain of the operator N is:

$$\begin{aligned} \|N\|_\infty &= \sup_{\Delta F \in L_\infty} \frac{\|N(\Delta F)\|_\infty}{\|\Delta F\|_\infty} = \sup_{\Delta F_{max} > 0} \frac{\sup_{\|\Delta F\|_\infty = \Delta F_{max}} \|N(\Delta F)\|_\infty}{\Delta F_{max}} \\ &= \sup_{\Delta F_{max} > 0} \frac{\sup_{\|\Delta F\|_\infty = \Delta F_{max}} \sup_{t > 0} |(N)(\Delta F)(t)|}{\Delta F_{max}} \\ &= \sup_{\Delta F_{max} > 0} \frac{\sup_{\substack{\|\Delta F\|_\infty = \Delta F_{max} \\ t > 0}} |\Delta T(t, \Delta F)|}{\Delta F_{max}} \\ &= \sup_{\Delta F_{max} > 0} \frac{|\Delta T(\infty, -\Delta F_{max})|}{\Delta F_{max}} \end{aligned} \quad (C16)$$

After some algebraic manipulations Eq. C16 yields

$$\|N\|_\infty = \sup_{\Delta F_{max} > 0} \frac{|\Delta T(\infty, -\Delta F_{max})|}{\Delta F_{max}} = \sup_{\Delta F_{max} > 0} \frac{|T_i - T_s|}{\left|F_s - \Delta F_{max} + \dfrac{UA}{\rho c_p}\right|} = \infty$$

$$\text{for } \Delta F_{max} = F_s + \frac{UA}{\rho c_p} > 0 \quad (C17)$$

Appendix D: Linearization Operator Gain and Incremental Gain

Theorem 2. Let N: $L^m_{pe} \rightarrow L^n_{pe}$ and Lu_o: $L^m_{pe} \rightarrow L^n_{pe}$ be an unbiased operator and its linearization around the trajectory u_o, respectively. Let also W be a *convex* subset of L^m_p. Then, for $p \in [1, \infty]$

$$\|N\|_{\Delta pW} \hat{=} \sup_{\substack{u_1,u_2 \in W \\ u_1 \neq u_2}} \frac{\|Nu_1 - Nu_2\|_p}{\|u_1 - u_2\|_p} = \sup_{u_o \in W} \|Lu_o\|_p$$

Proof. Based on Eq. 8 we have that

$$\lim_{\|u\|_p \to 0} \frac{\|N(u_o + u) - Nu_o - L_{u_o}u\|_p}{\|u\|_p} = 0 \quad \text{(D1)}$$

It also holds (triangle inequality) that

$$\frac{\|N(u_o + u) - Nu_o - L_{u_o}u\|_p}{\|u\|_p} \geq \frac{\|L_{u_o}u\|_p}{\|u\|_p} - \frac{\|N(u_o + u) - Nu_o\|_p}{\|u\|_p} \quad \text{(D2)}$$

Taking limits and using Eq. D1 we get

$$\lim_{\|u\|_p \to 0} \frac{\|N(u_o + u) - Nu_o\|_p}{\|u\|_p} \geq \lim_{\|u\|_p \to 0} \frac{\|L_{u_o}u\|_p}{\|u\|_p} \quad \text{(D3)}$$

By substituting $u_o + u$ by u_1 and u_o by u_2, and since L_{u_o} is linear we can write

$$\lim_{\|u\|_p \to 0} \frac{\|N(u_o + u) - Nu_o\|_p}{\|u\|_p} \geq \frac{\|L_{u_o}v\|_p}{\|v\|_p} \; \forall \, v \in L^m_p \Longrightarrow$$

$$\Longrightarrow \sup_{\substack{u_1 \in W \subset L^m_p \\ u_1 \neq u_2}} \frac{\|N(u_1) - Nu_2\|_p}{\|u_1 - u_2\|_p} \geq \sup_{u \in L^m_p - \{0\}} \frac{\|L_{u_o}v\|_p}{\|v\|_p} = \|L_{u_o}\| \Longrightarrow$$

$$\Longrightarrow \sup_{\|u_2 \in W} \sup_{\substack{u_1 \in W \\ u_1 \neq u_2}} \frac{\|N(u_1) - Nu_2\|_p}{\|u_1 - u_2\|_p} \geq \sup_{\|u_2 \in W} \|L_{u_o}\| \quad \text{(D4)}$$

where V is a neighborhood of 0, excluding 0. Next we prove that for $\alpha \in \boldsymbol{R}$ and $u_1 \neq u_2$ we have

$$\frac{dN(u_1 + \alpha(u_2 - u_1))}{d\alpha} = L_{u_1+\alpha(u_2-u_1)}(u_2 - u_1) \quad \text{(D5)}$$

Indeed,

$$\lim_{h \to 0} \left\| \frac{N(u_1 + (\alpha + h)(u_2 - u_1)) - N(u_1 + \alpha(u_2 - u_1))}{h} - L_{u_1+\alpha(u_2-u_1)}(u_2 - u_1) \right\|_p =$$

$$= \lim_{h \to 0} \frac{\|N(u_1 + \alpha(u_2 - u_1) + h(u_2 - u_1)) - N(u_1 + \alpha(u_2 - u_1)) - L_{u_1+\alpha(u_2-u_1)}h(u_2 - u_1)\|_p}{\|h(u_2 - u_1)\|_p} \|u_2 - u_1\|_p = 0\|u_2 - u_1\|_p = 0.$$

Thus we have:

$$Nu_2 - Nu_1 = \int_0^1 \frac{dN}{d\alpha}(u_1 + \alpha(u_2 - u_1))d\alpha \overset{\text{(D5)}}{=} \int_0^1 L_{u_1+\alpha(u_2-u_1)}(u_2 - u_1)\, d\alpha \Longrightarrow$$

$$\Longrightarrow \|Nu_2 - Nu_1\|_p \leq \int_0^1 \|L_{u_1+\alpha(u_2-u_1)}\|_p \cdot \|u_2 - u_1\|_p \, d\alpha \leq$$

$$\leq \|u_2 - u_1\|_p \int_0^1 \sup_{\alpha \in [0,1]} \|L_{u_1+\alpha(u_2-u_1)}\|_p \, d\alpha = \|u_2 - u_1\|_p \sup_{\alpha \in [0,1]} \|L_{u_1+\alpha(u_2-u_1)}\|_p \Longrightarrow$$

$$\Longrightarrow \frac{\|Nu_1 - Nu_2\|_p}{\|u_1 - u_2\|_p} \leq \sup_{\alpha \in [0,1]} \|L_{u_1+\alpha(u_2-u_1)}\|_p \Longrightarrow$$

$$\Longrightarrow \sup_{\substack{u_1 \in V_1 \\ u_2 \in V_2 \\ u_1 \neq u_2}} \frac{\|Nu_1 - Nu_2\|_p}{\|u_1 - u_2\|_p} \leq \sup_{\substack{u_1 \in V_1 \\ u_2 \in V_2}} \sup_{\alpha \in [0,1]} \|L_{u_1+\alpha(u_2-u_1)}\|_p$$

$$= \sup_{\substack{u_1 \in V_1 \\ u_2 \in V_2 \\ \alpha \in [0,1]}} \|L_{u_1+\alpha(u_2-u_1)}\|_p = \sup_{u_o \in co(V_1 \cup V_2)} \|L_{u_o}\|_p \quad \text{(D6)}$$

Now pick $V_1 = V_2 = W$ convex. Hence, by Eq. D6

$$\sup_{\substack{u_1 \in W \\ u_2 \in W}} \frac{\|N(u_o + u) - Nu_2\|_p}{\|u_1 - u_2\|_p} \leq \sup_{u_o \in W} \|L_{u_o}\|_p \tag{D7}$$

(D4) and (D7) $\Longrightarrow$

$$\sup_{\substack{u_1, u_2 \in W \\ u_1 \neq u_2}} \frac{\|Nu_1 - Nu_2\|_p}{\|u_1 - u_2\|_p} = \sup_{u_o \in W} \|L_{u_o}\|_p, \quad W \text{ convex} \quad \text{QED.}$$

Appendix E: Incremental Gain and Nonsmooth Optimal Control

Proof. The unique solution of Eq. 25 is

$$\Delta x(t) = \theta(t, 0)\Delta x(0) + \int_0^t \theta(t, \tau)B(\tau)u(\tau)\, d\tau \Longrightarrow$$

$$\Longrightarrow y(t) = \int_0^t C\theta(t, \tau)B(\tau)u(\tau)\, d\tau \triangleq \int_0^t W(t, \tau)u(\tau)\, d\tau$$

where $\theta(t, \tau)$ is the state transition matrix defined by

$$\frac{\partial \theta(t, \tau)}{\partial \tau} = -\theta(t, \tau)A(\tau), \theta(t, t) = I$$

By applying lemma Eq. 26 we get:

$$\|L_{u_o}\|_\infty = \sup_{t \in [0, \infty)} \int_0^t \|C\theta(t, \tau)B(\tau)\|_i\, d\tau$$

Application then of theorem 2 yields

$$\|N\|_{\Delta\infty W} = \sup_{u_o \in W} \|L_{u_o}\|_\infty = \sup_{u_o \in W} \sup_{t \in (0, \infty)} \int_0^t \|C\theta(t, \tau)B(\tau)\|_i\, d\tau =$$
$$= \sup_{t \in [0, \infty)} \sup_{u_o \in W} \int_0^t \|C\theta(t, \tau)B(\tau)\|_i\, d\tau =$$
$$= \sup_{t \in [0, \infty)} \left[-\inf_{u_o \in W} \int_0^t - \|C\theta(\tau)B(\tau)\|_i\, d\tau \right] \quad \text{QED}$$

Appendix F: L_∞ Incremental Set Gain of a Heater

$$(29) \Longrightarrow H(\Delta F(\tau)) = \left[p_1(\tau)\Phi(\tau)\frac{1}{V} - p_2(\tau)\frac{1}{V}\left(\Delta T(\tau) + \frac{UA(T_c - T_i)}{\rho c_p F_s + UA}\right)\right]\Delta F(\tau) + E(\tau)$$

$$\overset{(11)}{\Longrightarrow} \Delta F(\tau) = \begin{cases} -\epsilon, & \text{if } sgn\left[p_1(\tau)\Phi(\tau)\frac{1}{V} - p_2(\tau)\frac{1}{V}\left(\Delta T(\tau) + \frac{UA(T_c - T_i)}{\rho c_p F_s + UA}\right)\right] = 1 \\ \delta, & \text{if } sgn\left[p_1(\tau)\Phi(\tau)\frac{1}{V} - p_2(\tau)\frac{1}{V}\left(\Delta T(\tau) + \frac{UA(T_c - T_i)}{\rho c_p F_s + UA}\right)\right] = -1 \end{cases}$$

$$(28) \Longrightarrow \Phi(\tau) = \exp\left[\int_t^\tau \frac{1}{V}\left(\Delta F(s) + \frac{UA}{\rho c_p} + F_s\right) ds\right]$$
$$= \exp\left[\int_t^\tau \frac{1}{V}\Delta F(s)\, ds + \frac{1}{V}(\tau - t)\left(\frac{UA}{\rho c_p} + F_s\right)\right], \forall \tau \in (0, t] \tag{F1}$$

Therefore it holds that

$$\Phi(\tau) > 0, \quad \forall \tau \in (0, t]$$

In addition, from Eq. 18, we have

$$\Delta T(\tau) + \frac{UA(T_c - T_i)}{\rho c_p F_s + UA} = T(\tau) - T_i > 0, \quad \forall \tau \in (0, t]$$

for fundamental physical reasons. (It should be noted that elaborate but straightforward mathematical analysis yields the same result.) Thus Eqs. 30 and 33 hold everywhere on $(0, t]$, and combined with Eq. 36, yield:

$$p_1(\tau) = \int_0^\tau \exp\left[-\int_s^\tau \frac{1}{V}\Delta F(r)\, dr - \frac{1}{V}(\tau - s)\left(\frac{UA}{\rho c_p} + F_s\right)\right]$$
$$\cdot \frac{1}{V}\left(\Delta T(s) + \frac{UA(T_c - T_i)}{\rho c_p F_s + UA}\right) ds$$

$$p_2(\tau) = (\tau - t)\frac{1}{V}\exp\left[\int_t^\tau \frac{1}{V}\Delta F(r)\, dr + \frac{1}{V}\left(\frac{UA}{\rho c_p} + F_s\right)(\tau - t)\right]$$

Hence, by Eq. F1, we get that

$$sgn\left[p_1(\tau)\Phi(\tau)\frac{1}{V} - p_2(\tau)\frac{1}{V}\left(\Delta T(\tau) + \frac{UA(T_c - T_i)}{\rho c_p F_s + UA}\right)\right]$$
$$= sgn\left\{\frac{1}{(V)^2}\int_0^\tau \exp\left[\int_t^s \frac{1}{V}\Delta F(r)\,dr + \frac{1}{V}(s-t)\right.\right.$$
$$\left.\cdot\left(\frac{UA}{\rho c_p} + F_s\right)\right]\cdot\left(\Delta T(s) + \frac{UA(T_c - T_i)}{\rho c_p F_s + UA}\right)ds$$
$$-\frac{1}{(V)^2}(\tau - t)\exp$$
$$\cdot\left[\int_t^\tau \Delta F(r)\,dr + \frac{1}{V}\left(\frac{UA}{\rho c_p} + F_s\right)(\tau - t)\right]$$
$$\left.\cdot\left(\Delta T(\tau) + \frac{UA(T_c - T_i)}{\rho c_p F_s + UA}\right)\right\},\quad \tau \in (0, t] \tag{F2}$$

Claim. The value of Eq. F2 is 1. Indeed

$$\Delta T(s) + \frac{UA(T_c - T_i)}{\rho c_p F_s + UA} = T(s) - T_i > 0 \longrightarrow$$
$$\longrightarrow \frac{1}{V^2}\int_0^\tau \exp\left[\int_t^s \frac{1}{V}\left(\Delta F(r) + \frac{UA}{\rho c_p} + F_s\right)dr\right]$$
$$\cdot\left(\Delta T(s) + \frac{UA(T_c - T_i)}{\rho c_p F_s + UA}\right)ds > 0 \tag{F3}$$

and, since $t > \tau$

$$\frac{1}{V^2}(t - \tau)\exp\left[\int_t^\tau \frac{1}{V}\left(\Delta F(r) + \frac{UA}{\rho c_p} + F_s\right)dr\right]$$
$$\cdot\left(\Delta T(\tau) + \frac{UA(T_c - T_i)}{\rho c_p F_s + UA}\right) > 0 \tag{F4}$$

Combining Eq. F4 with Eqs. F3 and F2 we get that

$$sgn\left[p_1(\tau)\Phi(\tau)\frac{1}{V} - p_2(\tau)\frac{1}{V}\left(\Delta T(\tau) + \frac{UA(T_c - T_i)}{\rho c_p F_s + UA}\right)\right]$$
$$= 1 \longrightarrow \Delta F(\tau) = -\epsilon,\quad \tau \in (0, t]$$

Then it holds:

$$\sup_{t\in[0,\infty)}\left[-\min_{\Delta F\in W}\int_0^t -\left|\Phi(\tau)\frac{1}{V}\left(\Delta T(\tau) + \frac{UA(T_c - T_i)}{\rho c_p F_s + UA}\right)\right|d\tau\right]$$
$$= \sup_{t\in[0,\infty)}\frac{1}{\dfrac{UA}{\rho c_p} + F_s - \epsilon}\left(1 - \exp\left[-\frac{1}{V}\left(\frac{UA}{\rho c_p} + F_s - \epsilon\right)t\right]\right)$$
$$\cdot\frac{(T_c - T_i)}{UA + \rho c_p(F_s - \epsilon)} - \frac{1}{V}\frac{\epsilon UA(T_c - T_i)t}{(\rho c_p F_s + UA)\left(\dfrac{UA}{\rho c_p} + F_s - \epsilon\right)}$$
$$\cdot\exp\left[-\frac{1}{V}\left(\frac{UA}{\rho c_p} + F_s - \epsilon\right)t\right]$$
$$\longrightarrow \|N\|_{\Delta\infty W} = \frac{(T_c - T_i)\dfrac{UA}{\rho c_p}}{\left(\dfrac{UA}{\rho c_p} + F_s - \epsilon\right)^2}$$

This is the ∞ *set-incremental gain* of the operator N describing the heater.

Appendix G

Proof.

$$\sup_{u_1,u_2\in W}\frac{\|Nu_1 - Nu_2\|_p}{\|u_1 - u_2\|_p} \geq$$
$$\geq \sup_{u_2 - 0\subset W}\frac{\|Nu_1 - N(0)\|_p}{\|u_1\|_p} \tag{G1}$$

Since N is assumed to be unbiased we have $N(0) = 0$ and Eq. G1 yields

$$\sup_{u_1,u_2\in W}\frac{\|Nu_1 - Nu_2\|_p}{\|u_1 - u_2\|_p} \geq \sup_{u\in W}\frac{\|Nu\|_p}{\|u\|_p}$$
$$\longrightarrow \|N\|_{\Delta pW} \geq \|N\|_{pW}\quad \text{QED}$$

Notation

$\forall$ = for every
$\in$ = belongs to
$\notin$ = does not belong to
$\subset$ = subset of
$\cup$ = union
$\langle\cdot,\cdot\rangle$ = inner product
$\|\cdot\|$ = norm of a vector
A = area of heat exchange
co = convex hull
C_A = concentration of A
c_p = specific heat
$d_C(x)$ = distance of x from C
E = activation energy
F = flow rate
$\nabla f(x)$ = gradient of f at x
$f^\circ(x; v)$ = generalized directional derivative of f: $R^n \rightarrow R$ at x in the direction v
H = Hamiltonian
ΔH_R = heat of reaction
$\|\cdot\|_i$ = induced norm of a matrix
inf = largest lower bound of an ordered set
lim sup = yields the largest accumulation point of a set
L_p^n = Banach space of p-integrable n-vector-valued functions
L_{pe}^n = extended Banach space of p-integrable n-vector-valued functions
N = nonlinear operator
$\|N\|_p$ = dynamic p gain of N
$\|N\|_{\Delta p}$ = dynamic incremental p gain of N
$\|N\|_{pW}$ = the dynamic p gain of N over the set W
$\|N\|_{\Delta pW}$ = the dynamic incremental p gain of N over the set W
$N_C(x)$ = normal cone to $C \subset R^n$ at x
$\|\cdot\|_p$ = p norm of a vector-valued function
$\|\cdot\|_{pe}$ = extended p norm of a vector-valued function
p = costate vector
R = gas constant
$\mathbf{R}$ = set of real numbers
$\mathbf{R}^n$ = space of n-dimensional real vectors
$\mathbf{R}^{l\times l}$ = space of $l \times l$ real matrices
$sgn(\cdot)$ = sign of
sup = least upper bound of an ordered set
T = temperature
$T_C(x)$ = tangent cone to $C \subset R^n$ at x
U = heat transfer coefficient
u = input vector
V = volume
x = Cartesian product
x = state vector
y = output vector

Greek letters

α = kinetic constant
$\theta(t, \tau)$ = state transition matrix
$\Phi(\tau) = \theta(t, \tau)$
ϕ = function $\mathbf{R}^l \times \mathbf{R}^m \rightarrow \mathbf{R}^l$
λ = constant equal to 0 or 1
ρ = density
$\partial f(x)$ = generalized gradient of $f: \boldsymbol{R}^n \rightarrow \boldsymbol{R}$ at x

Subscripts

c = coil
i = input
s = steady state

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